Puzzle:

Two robots, Aaron and Erin, have made it to this year's final! Initially they are situated at the center of a unit circle. A flag is placed somewhere inside the circle, at a location chosen uniformly at random. Once the flag is placed, Aaron is able to deduce its distance to the flag, and Erin is only able to deduce its direction to the flag. (Equivalently: if (r, θ) are the polar coordinates of the flag's location, Aaron is told r and Erin is told θ.)

Both robots are allowed to make a single move after the flag is placed, if they wish. Any move they make is without knowledge of what the other robot is doing. (And they may not move outside the circle.)

Whichever robot is closer to the flag after these moves captures the flag and is declared the winner!

During the preliminaries it was discovered that Erin is programmed to play a fixed distance along the detected angle θ. Assuming otherwise optimal play by both robots, can you determine the probability that Aaron will win? (Please express your answer to 10 decimal places.)

Site Link: https://www.janestreet.com/puzzles/robot-capture-the-flag-index/

Solution: The flag position (r, θ) is drawn from a uniform distribution over the unit circle. Let r_e be the fixed distance that Erin will move toward the flag.

Before the start of the game Aaron is told r, but assume also that Aaron learned r_e somehow. Then the optimal distance for Aaron to move is r_a^* given by

$$\begin{aligned} r_a^* \,|\, r, r_e &= \begin{cases} \sqrt{r^2 - (r - r_a)^2} & \text{if } r \geq |r - r_a| \\ 0 & \text{if } r < |r - r_a| \end{cases} \\ &= \begin{cases} \sqrt{2r_a r - r_a^2} & \text{if } r \geq r_a/2 \\ 0 & \text{if } r < r_a/2 \end{cases} \end{aligned} \tag{1}$$

To understand these relations, observe that $r < r_a/2$ when the flag is placed closer to the origin than to Erin. In this case, Aaron can always win by choosing to move a distance 0 at the start of the game. In the case where $r \geq r_a/2$, Aaron's goal is to move the distance which creates the larges overlap (radians) with a circle centered at $(r, 0)$ and where the radius is $|r - r_e|$.

In the case where Aaron moves r_a^* in the game, the probability of Aaron winning the game (ending closer to the flag than Erin) is given by

$$P[\,win \,|\, r, r_e] = \begin{cases} \frac{\arcsin\left(|r - r_e|/r\right)}{\pi} & \text{if } r \geq r_e/2 \\ 1 & \text{if } r < r_e/2 \end{cases} \tag{2}$$

Considering that r is chosen from a uniform distribution over a unit disc, we have that

$$pdf(r) = pdf(r \,|\, \theta) = 2r \tag{3}$$

and

$$\begin{aligned} P[\,win \,|\, r_e] &= \int_0^1 P[\,win \,|\, r, r_e]\, pdf(r)\, \partial r \\ &= \int_0^{r_e/2} P[\,win \,|\, r, r_e]\, pdf(r)\, \partial r + \int_{r_e/2}^1 P[\,win \,|\, r, r_e]\, pdf(r)\, \partial r \\ &= \int_0^{r_e/2} 2r\, \partial r + \frac{2}{\pi} \int_{r_e/2}^1 r \cdot \arcsin\left(|r - r_e|/r\right) \partial r \\ &= \frac{r_e^2}{4} + \frac{2}{\pi} \int_{r_e/2}^{r_e} r \cdot \arcsin\left(\frac{r_e - r}{r}\right) \partial r + \int_{r_e}^1 r \cdot \arcsin\left(\frac{r - r_e}{r}\right) \partial r \end{aligned} \tag{4}$$

Reducing further we end with

$$P[\,win\,|\,r_e] = \frac{r_e^2}{4} + \frac{2}{\pi}\left(\frac{r_e^2}{16} + \frac{1}{2}\arcsin(1-r_e) - \frac{1+r_e}{6}\sqrt{r_e(2-r_e)}\right) \tag{5}$$

Thus, if Aaron was able to surmise r_e before the game and moved optimally, the probability Aaron would win would be (5).

Now consider the Game from Erin's perspective. She would choose to minimize this probability by solving $\frac{\partial P}{\partial r_e} = 0$, where

$$\frac{\partial P}{\partial r_e} = \frac{r_e}{4} + \frac{8r_e}{3\pi} + \frac{2}{\pi}\frac{(r_e+1)(r_e-2)}{3\sqrt{r_e(2-r_e)}}. \tag{6}$$

At this point, I was unable to reduce further by hand, and used a numerical solver to compute r_e satisfying $\frac{\partial P}{\partial r_e}(r_e) = 0$. The equation is solved by $r_e = 0.50130699421275304317$ and, in this case, the probability of Aaron winning is $P(win) = 0.16618648647400852125$.