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Lyapunov Differential Equation Hierarchy and Polynomial Lyapunov Functions for Switched Implicit Systems

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Abstract— **This paper investigates stability analysis for implicit, switched linear systems using homogeneous Lyapunov functions (HLF). HLFs of increasing degree are constructed through an outer-product, lifting transformation of the state vector to higher dimensions. This paper presents linear matrix inequalities sufficient conditions for asymptotic stability of these systems based on HLFs. A method is provided to search for Lyapunov functions by incrementally increasing the degree of the homogeneous Lyapunov functions. To address the dimensional growth of the problem space incurred by the lifting transform, a method for dimensional reduction is derived.**

I. INTRODUCTION

Modeling systems with implicit equations generalizes explicit systems by admitting both differential and algebraic equations. Implicit systems arise naturally in modeling Hamiltonian systems [1], dissipative systems [2], and robotic systems [3]. Implicit systems of the linear form are studied in [4] and [5]. The work [6] extends results pertaining to implicit linear systems to implicit polynomial, non-linear systems. Implicit systems allow for a more general model, but at the cost of an enlarged problem space [7].

The works [8] and [9] develop stability analysis using homogeneous polynomial Lyapunov functions. Abate *et al.* in [10], [11], and [12] extend these results using non-quadratic Lyapunov functions generated by outer-product multiplication of the state vector with itself to generate a higher-degree, homogeneous vector.

Abate's innovation in [10] is to simplify the search for Lyapunov functions by restricting the search space to homogeneous Lyapunov functions (HLF) implicitly defined via quadratic Lyapunov functions of larger linear systems obtained by lifting the original system. This approach is also possible with sum of squares programming (SOSP) [13], though, in general, the SOSP search is not restricted to homogeneous functions, rather uses a set of monomials of degree $\leq d$ which is useful for nonlinear systems. However, for linear systems and other homogeneous systems of degree 1, homogeneous Lyapunov functions are sufficient. By limiting the Lyapunov functions to strictly homogeneous functions, we formulate the search using Linear Matrix Inequalities (LMIs) [14].

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The concept of simplifying an optimization problem by augmenting the problem space with additional variables is described in [15] and illustrated geometrically by Fawzi *et al.* [16], for example an octagon with eight edges in $\mathbb{R}^2$ is lifted to a polyhedron of six faces in $\mathbb{R}^3$, where the edges or faces express the constraints of the problem. There are two primary mechanisms exploited in lifting: The first is a reduction in the problem complexity by reducing constraints. The second is an increase in the number of variables that corresponds to geometric objects with a greater number of principal axes than in ellipsoids; these objects better approximate the positive invariant sets of a given system. The main motivation for this paper is to determine a means of lifting Lyapunov stability problems to higher degree to simplify the problem representation and computation.

This work provides a method for constructing HLFs for implicit, switched linear systems that searches for polynomial Lyapunov functions for implicit systems over a convex space. The search is implemented by successively lifting to homogeneous Lyapunov functions of increasing degree. An upper bound on the degree of the HLF is specified to ensure eventual termination of the search.

The machinery for HLFs is extended in two directions. First, we generalize HLFs to support implicit systems. Second, we provide a method for reducing the dimensionality of the problem space systems of order $n > 2$.

This paper is organized in the following way. Section II establishes notation for Kronecker tensor product. Section III presents preliminaries on implicit systems and stability criteria. The main result, established in Section IV, is the construction of homogeneous Lyapunov functions for implicit systems and the LMI criteria for asymptotic stability of switched linear systems. We conclude with an algorithm that eliminates repetition in the state vector, thereby reducing the dimension of the lifted systems, homogeneous Lyapunov functions, and LMIs certifying stability.

II. NOTATION

We denote the non-negative and positive integers by $\mathbb{Z}_+$ and $\mathbb{Z}_{++}$, respectively. The set of symmetric positive-definite $n \times n$ matrices is denoted $\mathbb{S}^n_{++} \subset \mathbb{R}^{n \times n}$.

Given $F \in \mathbb{R}^{n \times m}$, we denote by $F^{\otimes d} \in \mathbb{R}^{n^d \times m^d}$ the dth *Kronecker Power* of F, which is defined inductively [17, *def. 4.2.2*], with base case $F^{\otimes 0} := 1$, as

$$F^{\otimes d} := F \otimes F^{\otimes d-1}, \qquad d \geq 1.$$

Definition 1. Given $E \in \mathbb{R}^{n\times n}$, $F \in \mathbb{R}^{n\times m}$, and $G \in \mathbb{R}^{m\times m}$ we define the dth *Kronecker Factor* of F by (E, G) for $1 \le k \le d$ as

$$\{E, F, G\}_k^{\otimes d} := E^{\otimes k-1} \otimes F \otimes G^{\otimes d-k}$$

where F is the kth factor from the left. ■

For example, $\{E, F, G\}_1^{\otimes 2} = F \otimes G$ and $\{E, F, G\}_2^{\otimes 2} = E \otimes F$.

Let $I_p \in \mathbb{R}^{p\times p}$ be the identity matrix of dimension $p \ge 1$. Two definitions for the sums of Kronecker factors are

$$F^{E\oplus d} := \textstyle\sum_{k=1}^{d} \{E, F, E\}_k^{\otimes d}$$
$$F^{\oplus d} := F^{I_p \oplus d}.$$

An identity for the product of Kronecker factors is

$$F^{\otimes d} \equiv \textstyle\prod_{k=1}^{d} \{I_m, F, I_n\}_k^{\otimes d}.$$

III. Preliminaries

An *implicit switched linear system* with state $x \in \mathbb{R}^n$ is given by

$$E(t)\dot{x} = A(t)x \tag{1}$$

where the pair $(E(t), A(t)) \in \mathbb{R}^{p\times n} \times \mathbb{R}^{p\times n}$ with $p \ge n$, evolve nondeterministically in the finite set of switched modes

$$(E(t), A(t)) \in \{(E_1, A_1), \cdots, (E_M, A_M)\} \tag{2}$$

for all t. Let $\mathcal{I} := \{1, 2, \cdots, M\}$ denote the index set (E_i, A_i) of eq. (2).

When $p > n$, system (2) is comprised of *differential-algebraic equations* (DAE). In general, the last $p - n$ rows of E_i and A_i are zero and non-zero, respectively, giving rise to algebraic equations which serve as constraints. In this case, the first n rows of E_i and A_i, which are the usual differential equations (DE), describe the evolution of the system in time. The partitioning of $(E(t), A(t))$ into DE and DAEs is sufficiently nuanced that it will be explored in subsequent research.

A. Rank Conditions

In this paper, we assume $p = n$ and for all $i \in \mathcal{I}$ such that

$$\mathrm{rank}(A_i) = \mathrm{rank}(E_i) = n. \tag{3}$$

The full rank condition on E_i eliminates algebraic constraint equations and the full rank condition on A_i removes any zero dynamics. This may appear overly restrictive since premultiplying the inverse of E_i renders the *explicit* system dynamics

$$\dot{x} = E(t)^{-1}A(t)x. \tag{4}$$

However, this approach of explicitization can lead to numerical issues when E_i is poorly conditioned; this is the case for systems with "stiff" mass matrices.

B. Implicit Stability

In the literature, [18] and [19] establish criteria for *quadratically stabilizable* Lyapunov functions, where a single "common" function serves as a Lyapunov function for a switched linear system. Abate *et al.* [10] extends this to homogeneous Lyapunov functions. Here, we define stability of implicit switched linear systems (1) followed by a lemma establishing existence of a common Lyapunov function.

Definition 2 (*Implicit Common Lyapunov Stability*)**.** An implicit common Lyapuov function for a system is the mapping $V : \mathbb{R}^q \to \mathbb{R}$, for $q \in \mathbb{Z}_{++}$, such that the following statements hold

$$\begin{aligned} V(E_i x) &> 0 \\ \dot{V}(E_i x) = \langle \nabla_{E_i x} V,\, A_i x\rangle &< 0 \\ \forall x &\in \mathbb{R}^n \setminus \{\mathbf{0}\} \\ \forall i &\in \mathcal{I} \end{aligned} \tag{5}$$

where $\nabla_{E_i x} V$ is the gradient of V with respect to $E_i x$. ■

There are $2M$ inequality relations in (5). When $E_i = I$, Definition (2) reduces to the definition given in [10]. Mason *et al.* [8] and Ahmadi *et al.* [20] show that homogeneous Lyapunov functions exist for stable switched linear systems, though the degree of the Lypaunov function is not known a priori.

Lemma 1. The implicit switched linear system (1) is stable if an only if there exists homogeneous common Lyapunov function. ■

Proof. The systems (1) and (4) have equivalent stability properties, and a Lyapunov function for (4) will also be a Lyapunov function for (1). Applying the results of [8] and [20], the system (4) is asymptotically stable if an only if there exists homogeneous polynomial Lyapunov function certifying its stability and, in this instance, the stability of (1) can be shown with the same Lyapunov function. This completes the proof. □

IV. Main Results

The intent in this paper is to find homogeneous polynomial forms of degree $d > 2$ that serve as common Lyapunov functions for arbitrarily switched implicit linear systems. We start by lifting the system (1) and associated Lyapunov functions homogeneous forms of degree d and $2d$, respectively. Next, we establish necessary and sufficient conditions for asymptotic stability of the lifted system from the implicit switched linear systems. The main results are LMI conditions for asymptotic stability of the lifted system and transitively on the *original* implicit switched linear systems. Finally, we state the problem of searching for a viable common homogeneous Lyapunov function and present a solution.

A. Lifted System Dynamics

We construct lifts for systems and Lyapunov functions using the lifted system state $x \mapsto x^{\otimes d} \in \mathbb{R}^{n^d}$. Everything developed for the single system (E, A) can assumed to apply to the switched system (E_i, A_i).

Proposition 1 (*Lifted System*)**.** The lifting transform of system (1) to a homogeneous system of degree d is given by

$$E^{\otimes d}\tfrac{d}{dt}(x^{\otimes d}) = A^{E\oplus d}x^{\otimes d} \tag{6}$$

which is called the *lifted system*. ■

Proof. The approach of this proof is to show that when $x(t)$ is a solution to the original system, then $x(t)^{\otimes d}$ is a solution to the lifted system. First lift the term Ex to $(Ex)^{\otimes d}$. Then using the identify for the product of Kronecker Factors and taking the derivative, results in

$$\frac{\mathrm{d}}{\mathrm{d}t}(Ex)^{\otimes d} = \frac{\mathrm{d}}{\mathrm{d}t}\prod_{k=1}^{d}\{1, Ex, I_n\}_k^{\otimes d}.$$

Applying the chain rule gives

$$\frac{\mathrm{d}}{\mathrm{d}t}(Ex)^{\otimes d} = \sum_{j=1}^{d}\prod_{k=1}^{d}\begin{cases}\{1, Ex, I_n\}_k^{\otimes d}, & j \neq k\\ \{1, E\dot{x}, I_n\}_k^{\otimes d}, & j = k.\end{cases}$$

(See [21] for similarities.) Recognizing that $\frac{\mathrm{d}}{\mathrm{d}t}(E^{\otimes d}x^{\otimes d}) = E^{\otimes d}\frac{d}{dt}(x^{\otimes d})$ and substituting Ax for $E\dot{x}$ results in

$$\begin{aligned}E^{\otimes d}\tfrac{d}{dt}(x^{\otimes d}) &= \sum_{j=1}^{d}\prod_{k=1}^{d}\begin{cases}\{1, Ex, I_n\}_k^{\otimes d}, & j \neq k\\ \{1, Ax, I_n\}_k^{\otimes d}, & j = k\end{cases}\\ &= \sum_{j=1}^{d}\{Ex, Ax, Ex\}_j^{\otimes d}\\ &= \sum_{j=1}^{d}\{E, A, E\}_j^{\otimes d}x^{\otimes d}\\ &= A^{E\oplus d}x^{\otimes d}.\end{aligned}$$

This completes the proof. □

When $E = I$, the lifted system (6) reduces to $\frac{d}{dt}x^{\otimes d} = A^{I\oplus d}x^{\otimes d} =: A^{\oplus d}x^{\otimes d}$ as established by Abate *et al.* in [10].

Example 1. Consider the implicit system $e\dot{x} = ax \in \mathbb{R}$. In $\mathbb{R}$, the Kronecker product is scalar multiplication, so that $x^{\otimes 3} = x^3$, $e^{\otimes 3} = e^3$, and $a^{e\oplus 3} = (aee+eae+eea) = 3ae^2$. Now, the lifted system is expressed by

$$e^3\tfrac{d}{dt}(x^3) = 3ae^2x^3.$$

Further expanding the derivative on the left-hand side leads to $e^3(3x^2\dot{x}) = 3ae^2x^3$. Now, we take the opportunity to divide both sides by $3(ex)^2$, assuming $x \neq 0$, and we have returned to the linear system $e\dot{x} = ax$. This example exhibits equivalence between the linear and lifted system except at $x = 0$. This *projection* transformation from the lifted to the linear system becomes important for control design problems. ■

Example 2. Let $x = [y_1, y_2]^{\mathrm{T}} \in \mathbb{R}^2$. Consider the system

$$\begin{bmatrix}-1 & -3\\ -1 & 2\end{bmatrix}\begin{bmatrix}\dot{y}_1\\ \dot{y}_2\end{bmatrix} = \begin{bmatrix}-2 & 1\\ 3 & 1\end{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix}.$$

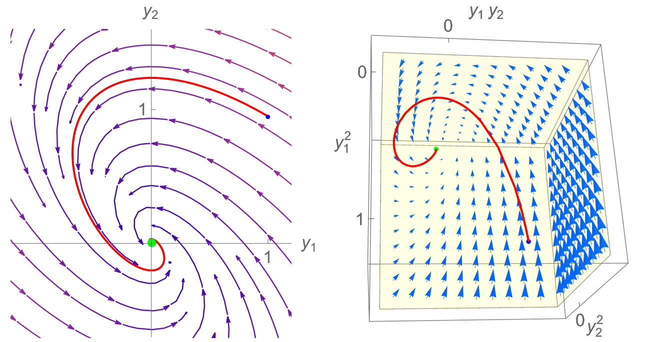

Fig. 1. (a) Phase diagram of $\mathbb{R}^2$ system with trajectory (red) from initial condition (blue dot) converging to equilibrium point (green dot). (b) Phase diagram of $\mathbb{R}^4$ projected to $\mathbb{R}^3$ system with trajectory (red) from initial condition (blue dot) converging to equilibrium point (green dot).

The lifted state $x^{\otimes 2} = [y_1^2,\ y_1y_2,\ y_2y_1,\ y_2^2]^{\mathrm{T}} \in \mathbb{R}^4$. Likewise, the matrices $E_2 = E^{\otimes 2}$ and $A_2 = A^{E\oplus 2}$ of the lifted system are computed as

$$E_2 = \begin{bmatrix}1 & 3 & 3 & 9\\ 1 & -2 & 3 & -6\\ 1 & 3 & -2 & -6\\ 1 & -2 & -2 & 4\end{bmatrix}\ A_2 = \begin{bmatrix}4 & 5 & 5 & 6\\ -1 & -5 & -10 & -1\\ -1 & -10 & -5 & -1\\ -6 & 5 & 5 & 4\end{bmatrix}.$$

Figure (1) shows the asymptotically stable phase diagram in $\mathbb{R}^2$ for the linear system. We note that the lifted state $x^{\otimes 2}$ repeats the component y_1y_2; the repeated component is removed by an elimination projection $W_2^+ : \mathbb{R}^4 \to \mathbb{R}^3$ in order to visualize the phase diagram in $\mathbb{R}^3$, shown in Figure (1). Likewise for the lifted system, the components of the lifted state y_2y_1, y_1^2, and y_2^2 converge to the origin $\mathbf{0}$. The elimination projection is discussed in section IV-D. ■

B. Lifted Lyapunov Function

Using the lifted state $x^{\otimes d}$, we can immediately construct lifted Lyapunov functions for a single system (E, A). However, for a switched system $(E(t), A(t))$, it is first necessary to show that a single, lifted Lyapunov function suffices for the switched system.

Consider lifted Lyapunov functions $V_d(x) : \mathbb{R}^{n^d} \to \mathbb{R}$ for homogeneous systems of degree d. Let $P_d \in \mathbb{S}_{++}^{n^d}$. Now we define the general *multilinear* Lyapunov function

$$V(w) := w^{\mathrm{T}}Pw \tag{7}$$

where $w \in \mathbb{R}^p$ and $P \in \mathbb{S}_{++}^p$. Now, we generalize eq. (7) to lifted Lyapunov functions for systems of degree d

$$V_d(x) := V\left((Ex)^{\otimes d}\right) \tag{8}$$

which is expanded to

$$V_d(x) = x^{\otimes d^{\mathrm{T}}}E^{\otimes d^{\mathrm{T}}}P_dE^{\otimes d}x^{\otimes d}. \tag{9}$$

For $d = 1$, eq. (9) reduces to the quadratic $V_1(x) = x^{\mathrm{T}}E^{\mathrm{T}}P_1Ex$ with $P_1 \in \mathbb{S}_{++}^n$. Asymptotic stability is determined on some domain $\mathcal{D} \subset \mathbb{R}^n$ by convergence of $V_d(x(t))$ to 0 as $t \to \infty$, which is the *negativity condition* in Definition

(2): $\dot{V}(Ex) < 0$. Taking the derivative of (9) and using $(Ex)^{\otimes d} = E^{\otimes d}x^{\otimes d}$ results in

$$\begin{aligned}&\dot{V}_d(x)\\&= (\tfrac{d}{dt}(Ex)^{\otimes d})^{\mathrm{T}} P_d (Ex)^{\otimes d} + ((Ex)^{\otimes d})^{\mathrm{T}} P_d \tfrac{d}{dt}(Ex)^{\otimes d}\\&= ((Ax)^{Ex\oplus d})^{\mathrm{T}} P_d (Ex)^{\otimes d} + ((Ex)^{\otimes d})^{\mathrm{T}} P_d (Ax)^{Ex\oplus d}.\end{aligned}$$

Recognizing that $(Ax)^{Ex\oplus d} = A^{E\oplus d}x^{\otimes d}$ and factoring out $x^{\otimes d}$ yields

$$\dot{V}_d(x) = x^{\otimes d^{\mathrm{T}}}(A^{E\oplus d^{\mathrm{T}}} P_d E^{\otimes d} + E^{\otimes d^{\mathrm{T}}} P_d A^{E\oplus d})x^{\otimes d}. \quad (10)$$

For $d = 1$, this result is given in [4] and [22]. An additional condition for stability is regularity of $(A^{E\oplus d}, E^{\otimes d})$, that is all roots for each pencil $\det(sE^{\otimes d} - A^{E\oplus d}) = 0$ are in open left-half plane. Regularity for implicit systems is the analogue of asymptotic stability for explicit systems.

C. Stability

To determine stability of the system (1), we first introduce the notion of equivalence of 2-norm for trajectories of the original and lifted systems. This allows us to show that asymptotic stability of the lifted system implies asymptotic stability of original system. This leads to the main results which are LMI tests for stability. Because these results are only sufficient, a procedure is developed that searches through HLFs of increasing degree d. Finally, we develop two approaches for reducing the dimension of the lifted systems for improved computational solutions.

Lemma 2. Let $v \in \mathbb{R}^p$ and $p \geq 0$. Then for $d \geq 1$

$$\|v^{\otimes d}\|_2 = \|v\|_2^d. \quad \blacksquare$$

Proof. Squaring the left-hand side of Lemma (2), followed by term-expansion via the inner-product, yields $\|v^{\otimes d}\|_2^2 = \langle v^{\otimes d}, v^{\otimes d}\rangle = (v^{\otimes d})^{\mathrm{T}}(v)^{\otimes d} = (v^{\mathrm{T}})^{\otimes d}(v)^{\otimes d} = (v^{\mathrm{T}}v)^{\otimes d}$. Because $v^{\mathrm{T}}v$ is a scalar, the Kronecker product becomes scalar multiplication such that $(v^{\mathrm{T}}v)^{\otimes d} = (v^{\mathrm{T}}v)^d = \|v\|_2^{2d}$. Taking the square root of the last term arrives at the right-hand side of Lemma (2). This completes the proof. □

Now we're ready to show that asymptotic stability (AS) of lifted and original systems implies AS of the other.

Proposition 2 (*Asymptotic Stability*)**.** The system (1) is AS if and only if the lifted system (6) is AS. ■

Proof. We show that the limit of every trajectory exists and is zero for both the necessary and sufficient conditions. The definition of asymptotic stability [23] says there exists $\delta > 0$ such that for every trajectory with initial conditions that satisfy $|x(0)| < \delta$, then $\lim_{t\to\infty} x(t) = \mathbf{0}$.

First we show sufficiency: AS of (6) ⇒ AS of (1). Let $y_d(t) = x(t)^{\otimes d}$. By AS, the $\lim_{t\to\infty} y_d(t) = \mathbf{0}$ for every trajectory starting at $y_d(x_0) \in D_d \subset \mathbb{R}^{n^d}$ where $x_0 \in \mathbb{R}^n$. Thus, $\|y_d(t)\|_2 \to 0$ for every trajectory. By Lemma (2), this implies implies $\|x(t)\|_2 \to 0$.

Next we show necessity: AS of (1) ⇒ AS of (6). Start with (1) AS, then $\lim_{t\to\infty} x(t) = \mathbf{0}$ for every trajectory starting at $x(0) \in D_1 \subset \mathbb{R}^n$. Again, by Lemma (2), we reverse the implications $x(t) \to \mathbf{0} \Rightarrow \|x(t)\|_2 \to 0 \Rightarrow \|y_d(t)\|_2 \to 0 \Rightarrow y_d(t) \to \mathbf{0}$ for every trajectory. This last expression shows that $\lim_{t\to\infty} y_d(t) = \mathbf{0}$. These complete the proof. □

From Proposition (2) we establish the main result: LMI conditions for asymptotic stability of the original system.

Theorem 1 (*Multilinearly Stable*)**.** *Let $V_d(y) = y^{\mathrm{T}}P_d y$ according to definition* (9)*, where $y \in \mathbb{R}^{n^d}$ and $P_d \in \mathbb{S}^{n^d}_{++}$. Then V_d multilinearly stabilizes system* (1) *when these statements hold*

$$P_d \succ 0 \quad (11)$$

$$A_i^{E_i\oplus d^{\mathrm{T}}} P_d E_i^{\otimes d} + E_i^{\otimes d^{\mathrm{T}}} P_d A_i^{E_i\oplus d} \prec 0 \quad (12)$$
$$\forall i \in \mathcal{I}.$$

V_d is called a common homogeneous Lyapunov function (CHLF) for lifted system (6). ■

Proof. Assume $V_d(y) = y^{\mathrm{T}}P_d y$ and $P_d \succ 0$, then $V_d(y) > 0$ for all $y \in \mathbb{R}^{n^d}\setminus\{\mathbf{0}\}$. Let $y = E_i x$, then $V_d(E_i x) > 0$ for all $i \in \mathcal{I}$. This establishes eq. (11) for all $i \in \mathcal{I}$. The conditions for the derivative $\dot{V}_d$ are given in (9). Then $\dot{V}_d < 0$ if and only if $A_i^{E_i\oplus d^{\mathrm{T}}} P_d E_i^{\otimes d} + E_i^{\otimes d^{\mathrm{T}}} P_d A_i^{E_i\oplus d} \prec 0, \ \forall i \in \mathcal{I}$. This establishes eq. (12) for all $i \in \mathcal{I}$. Therefore system (6) is AS, which implies that system (1) is AS by Proposition (2). This completes the proof. □

Remark 1. For the positive definite condition in Definition (2), there are M inequalities, one for each E_i, whereas, there is a single inequality in eq. (11). Though V_d was dependent on E_i in the definition, for the purposes of the LMI, E_i is no longer required. ■

Remark 2. The LMIs in Theorem (1) are not restricted to testing the space encompassing all possible the trajectories of $\{(E_i x)^{\otimes d}\} \subset \mathbb{R}^{n^d}$ of the lifted system (6), rather they test the larger, relaxed space $\mathbb{R}^{n^d}$. This relaxation of the problem space enables the use of LMIs to model and solve lifted stability problems despite introducing conservatism. However, this conservatism is overcome by the higher degree CHLFs; it is guaranteed to be smaller than that introduced by only using quadratic Lyapunov functions on system (1). A brief justification, without detail, is given by the set ordering $\min_{y\in\mathcal{D}} V(y) \leq \min_{x^{\otimes d}\in\mathcal{D}} V_1(x)$ induced by the inclusion $\{x^{\otimes d}\} \cap \mathcal{D} \subset \mathcal{D}$ for some compact $\mathcal{D} \subset \mathbb{R}^{n^d}$. ■

Lifted systems and lifted Lyapunov functions convert a quadratic problem to homogeneous problem which normally requires a sum of squares approach to solve. While Lemma (1) guarantees a CHLF for a stable system (1), there is no guarantee at which degree d we will find a solution. The following problem presents an approach finding $V_d(x)$.

Problem 1. Given system (1) known to be asymptotically stable, find a $P_d \in \mathbb{S}^{n^d}_{++}$, for the smallest $d \geq 1$ that satisfies eqs. (12). ■

For high degree d, evaluation of LMIs (11) and (12) becomes numerically impractical, as the SDP solvers suffer or fail due to problem size or conditioning issues. The next section explores methods of reducing problem space.

D. Stability on Reduced Space

In the case when the implicit system can be made explicit via E^{-1}, the dimensionality of the HLF search can be reduced as discussed in [10]. In the following we recast the results of [10] in terms of the implicit system (1) to accommodate this dimensionality reduction.

Consider a transformation of the lifted system statespace

$$W_d y_d := x^{\otimes d}. \tag{13}$$

Here, W_d is the *elimination matrix* as discussed in [10]. Next, we establish new stability LMIs similar to Theorem (1). Because W_d is a permutation matrix, the Moore-Penrose generalized inverse, denoted by W_d^+, also exists [24], such that $y_d = W_d^+ x^{\otimes d}$, where $W_d \in \{1,0\}^{n^d \times m_d}$ and $y_d \in \mathbb{R}^{m_d}$ is the resultant minimal vector, without repetition. Dimension $m_d = \binom{n+d-1}{d}$ is the number of multisets combinations.

Our desire is to reduce the dimension of P_d in (9); however, applying the reduction (13) directly results in no dimensional change of P_d. We must return to the system (1) and apply the reduction (13) which gives $E^{\otimes d} W_d \dot{y}_d = A^{E\oplus d} W_d y_d$. Multiplying both sides by $W_d^+ (E^{-1})^{\otimes d}$ results in the *explicit, reduced, lifted system*

$$\dot{y}_d = W_d^+ (E^{-1}A)^{\oplus d} W_d y_d \tag{14}$$

where $(E^{-1})^{\otimes d} A^{E\oplus d} = (E^{-1}A)^{E^{-1}E\oplus d} = (E^{-1}A)^{\oplus d}$. For convenience, let

$$\bar{A} = E^{-1}A \quad \text{and} \quad \bar{A}_i = E_i^{-1} A_i.$$

Now, we generate the reduced Lyapunov function from (7) such that

$$V(y_d) = y_d^{\mathrm{T}} \bar{P}_d y_d \tag{15}$$

where $\bar{P} \in \mathbb{S}_{++}^{m_d}$. Taking the derivative the (15) gives $\dot{V}(y_d) = \dot{y}_d^{\mathrm{T}} \bar{P}_d y_d + y_d^{\mathrm{T}} \bar{P}_d \dot{y}_d = y_d^{\mathrm{T}} (W_d^+ \bar{A}^{\oplus d} W_d)^{\mathrm{T}} \bar{P}_d y_d + y_d^{\mathrm{T}} \bar{P}_d (W_d^+ \bar{A} d_d^{\oplus W}) y_d$. This results in the reduced, lifted Lyapunov function derivative

$$\dot{V} = y_d^{\mathrm{T}} \Big((W_d^+ \bar{A}^{\oplus d} W_d)^{\mathrm{T}} \bar{P}_d + \bar{P}_d (W_d^+ \bar{A}^{\oplus d} W_d) \Big) y_d. \tag{16}$$

We generate the same asymptotic stability and multilinearly stabilizable results for this new, minimal space as we did for the linear transform.

Proposition 3 (*Asymptotic Stability for Projected Space*)**.** The system (1) is AS if and only if (14) is AS. ■

The proof is the same as Proposition (2), where $E = I$ and $A = \bar{A}$ and therefore is omitted.

Now, we restructure the main result from (13) for reduction of lifted spaces.

Theorem 2 (*Multilinearly Stable for Projected Space*)**.** *Let $V_d(y_d)$ be defined as in theorem* (1)*. Then $V_d(x)$ is a Lyapunov function for system* (1)

$$P \succ 0 \tag{17}$$

$$(W_d^+ \bar{A}_i^{\oplus d} W_d)^{\mathrm{T}} \bar{P}_d + \bar{P}_d (W_d^+ \bar{A}_i^{\oplus d} W_d) \prec 0 \tag{18}$$

$$\forall i \in \mathcal{I}. \quad ■$$

The proof is the same as Theorem (1), where we replace $E^{\otimes d} x^{\otimes d}$ with $E^{\otimes d} W_d y_d$, and therefore is omitted.

Remark 3. In order to get reduced equations, we paid the penalty of inversion of E. Notice that if for the original system, we can relax the constraint of full rank E, then the same solution will apply for our reduced lifted system. In subsequent paper, we will remove the full rank restriction. ■

A direct method for constructing W_d scans the vector $x^{\otimes d}$ and indexes repeated elements, keeping only unique values in y_d. The index and y_d are used to construct W_d.

A second method for constructing W_d is derived by computing recursively from W_{d-1}. The work [10] provides an algorithm for the case of $n = 2$, for all values of $d \in \mathbb{Z}_{++}$. This paper derives a similar recursive construction for the case $n = 3$, for all values of $d \in \mathbb{Z}_{++}$. There is no known general, recursive mapping $W_{n,d-1} \mapsto W_{n,d}$ for arbitrary n. In these constructions, we make use of the following definitions

$$0_{d,p} \in \{0\}^{d \times p} \qquad W_0 = 1 \in \mathbb{R}. \tag{19}$$

There is only one element in $\{0\}^{d \times p}$ which is the zeros matrix of size $d \times p$. We add an additional definitions $0_{d,0} \equiv \emptyset$ which simplifies the notation of the construction.

Proposition 4 (*Elimination Matrix $n = 2$ [10]*)**.** Consider the vector $x \in \mathbb{R}^n$ and $x^{\otimes d}$ for $n = 2$ and $d \in \mathbb{Z}_+$. The *elimination matrix* W_d that satisfies (13) is defined by

$$\begin{aligned} m_d &= d + 1 \\ W_0 &= 1 \\ W_{d+1} &= \left[\begin{array}{ccc} 0_{d,0} & W_{d,(1:m_d)} & 0_{d,1} \\ \hline 0_{d,1} & W_{d,(1:m_d)} & 0_{d,0} \end{array} \right]. \end{aligned} \quad ■$$

For $d = 0$ we get $W_1 = I_2$. The entries in each column do not have the same dimension because $0_{d,1}$ and $0_{d,0}$ are not the same size, generally.

Proposition 5 (*Elimination Matrix, $n = 3$*)**.** Consider the vector $x^{\otimes d}$, where $d \in \mathbb{Z}_+$ and $n = 3$. The matrix W_d that satisfies (13) is defined by

$$\begin{aligned} m_d &= \left(\frac{d+2}{d} \right) \\ W_0 &= 1 \\ W_{d+1} &= \left[\begin{array}{cc|cc|cc} 0_{d,0} & W_{d,(1:m_d)} & 0_{d,1} & W_{d,(m_d:m_d+k-1)} & & 0_{d,2} \\ \hline 0_{d,1} & W_{d,(1:1)} & 0_{d,1} & W_{d,(s_k:s_k+k-1)} & & 0_{d,1} \\ \hline 0_{d,2} & W_{d,(1:1)} & 0_{d,1} & W_{d,(s_k:s_k+k-1)} & k & 0_{d,0} \end{array} \right] \end{aligned}$$

where the middle 2 columns-group repeat k times left-to-right for $1 \leq k \leq d$. ■

For $d = 0$, there is no repetition, and we get $W_1 = I_3$. The entries in each column do not have the same dimension generally, and the matrix must be constructed for each row, column by column, going from left to right, including repetitions k. In the case where $e < s$, then we define $W_{d,(e<s)} \equiv \emptyset$.

The recursive construction of W_d is captured in algorithm (1). The storage at each step k is $m_k n^k$, which can be overbounded by $m_{d-1}n^{d-1} + m_d n^d$, which adds the storage of the next to last W_{d-1} to the final W_d.

Algorithm 1: *Reduction Matrix*

function `reductionMatrix`(d)**:**
  Input: $d \in \mathbb{Z}_{++}$ the degree of the multiset vector
  Output: $W_d \in \{0,1\}^{m_d \times n^d}$ reduction matrix
  % Recursively Build W_d
  if $d = 0$ **then** **return** $W_d \leftarrow 1$
  else $W_{d-1} \leftarrow$ `reductionMatrix`($d-1$)
  % Initialize
  $s \leftarrow 1;\quad e \leftarrow 1;\quad n \leftarrow 3$
  $W_{d,1} \leftarrow [\ 0_0 \quad W_{d-1}\ ]$ *% Row-group* $r = 1$: W_d
  for $r \leftarrow 2$ **to** n **do**
    $W_{d,r} \leftarrow [\ 0_{r-1} \quad W_{d-1,s:e}\ ]$
  % Iterations
  for $k \leftarrow 2$ **to** x **do**
    $s \leftarrow e + 1;\quad e \leftarrow s + k - 1$ *% Cols. (s)tart,(e)nd*
    $W_{d,1} \leftarrow [\ W_{d,1} \quad 0_1\ ]$
    for $r \leftarrow 2$ **to** n **do**
      $W_{d,r} \leftarrow [\ W_{d,r} \quad 0_1 \quad W_{d-1,s:e}\ ]$
  % Finalize
  for $r \leftarrow 1$ **to** n **do**
    $W_{d,r} \leftarrow [\ W_{d,r} \quad 0_{n-r+1}]$
  return W_d

V. CONCLUSION

In this paper, we generalized results for homogeneous Lyapunov functions to implicit systems, which includes LMI sufficient conditions for original and lifted systems. To reduce the exponential problem growth, we presented scan-and-index and recursive algorithms for constructing an elimination matrix transform. Areas for subsequent research include a comparison of HLF to structured SOS approaches such as AnySOS [25] and DSOS [26]. Furthermore, we treat implicit systems where E is not full rank with algebraic constraints. Also, it is worth investigating the recursive construction of the elimination matrix to see if it can be extended for $n > 3$.

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