Go Figure
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Go Figure is a set of challenging combinatorics problems.

- There are three puzzles, which appear individually on pages 2, 3 and 4. The solutions appear on page 5.

- If you want to challenge yourself, play the puzzles one at a time, in order, completing one without looking at the next (i.e., don't view page 3 until you've locked in your answer on page 2).

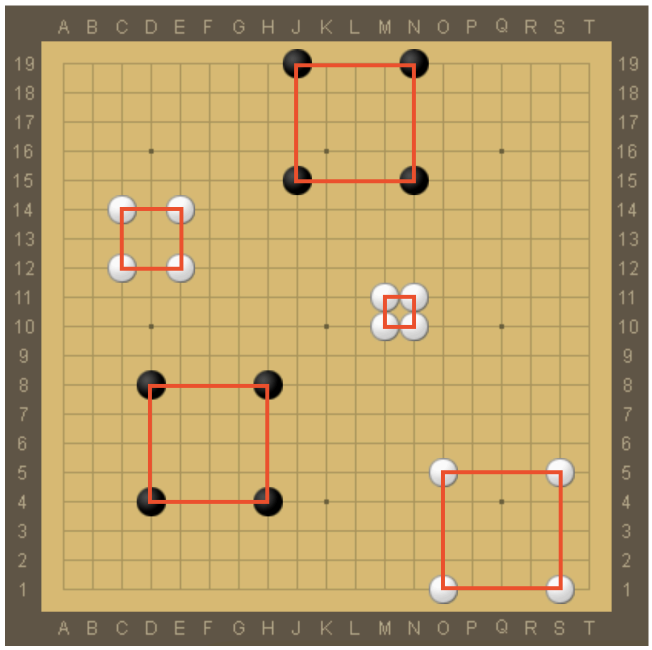
- If you want to take it easy, aren't sure where to start, or are having difficulty, consider playing the puzzles xin reverse order (i.e., look at all the puzzles at the start, then start on page 4, lock in an answer, and move to page 3).

- Fair warning – I don't shy away from trick questions.

The puzzles begin on the next page. Enjoy! – Matt

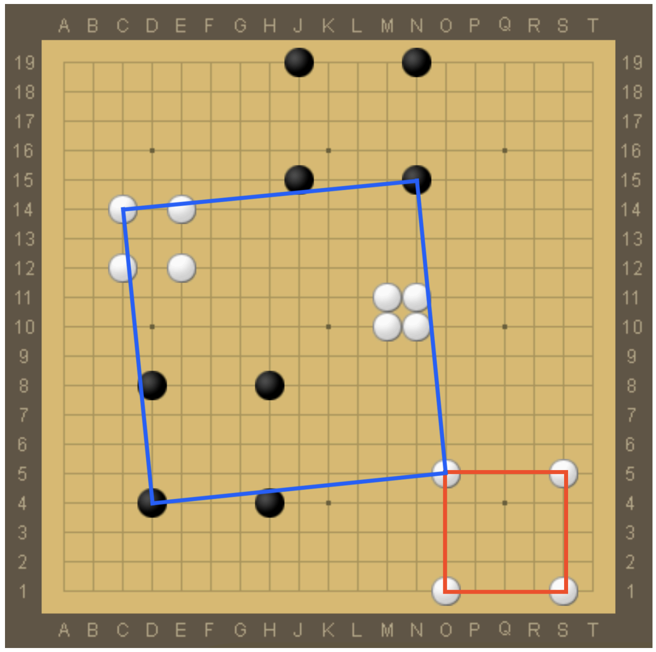
Intro: A standard Go Board looks like a 19x19 grid (19 vertices on a side). Players take turns placing stones on the grid's vertices, trying to make specific shapes, capture each other's pieces or reinforce their own.

Puzzle #1: How many squares can be formed on a 19×19 Go Board? A Go Board is a 19x19 grid and squares are formed by covering exactly four vertices with stones.



Number of Possible Squares: ________

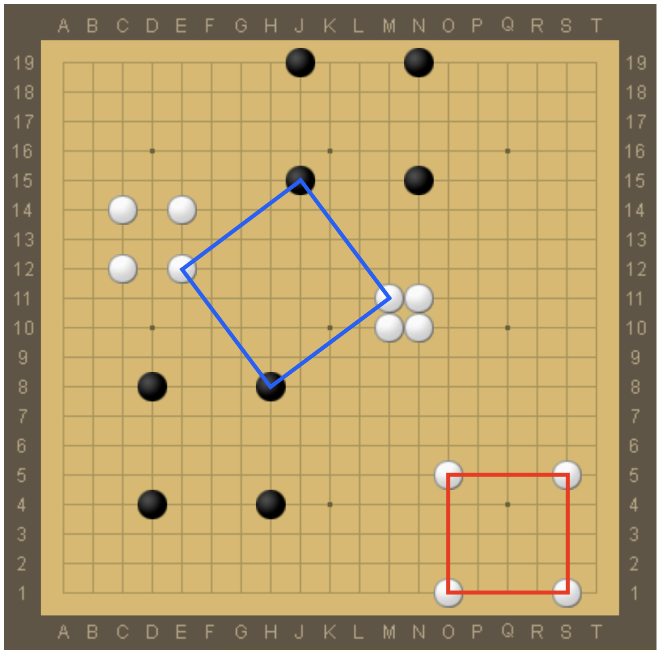
Puzzle #2: How many squares can be formed on a 19×19 Go Board? This time, only count squares with integer side-lengths.



Count squares like the **red** square, but not the **blue** square.

Number of Possible Squares with Integer Side-lengths: ________

Puzzle #3: How many squares can be formed on a 19×19 Go Board? This time, only count squares with sides parallel to the grid.



Count squares like the **red** square, but not the **blue** square.

Number of Possible Squares with Sides Parallel to Grid: ________

Solutions:

Puzzle #1: The total number of squares that can be formed on a 19x19 Go Board is

<u>10,830</u>.

Puzzle #2: Only counting squares with integer side-lengths, the total number of squares that can be formed on a 19x19 Go Board is

<u>2,455</u>.

Puzzle #3: Only counting squares whose sides are parallel to the grid's axis, the total number of squares that can be formed on a 19x19 Go Board is

<u>2,109</u>.

Explanation:

To compute these numbers, it's useful to start by counting squares whose sides are parallel to the grid's axis. Including such squares, there are

- 1 square with side-length 18,
- $4 = 2^2$ squares with side-length 17
- $\ldots$
- $324 = 18^2$ squares of side-length 1

Expanding the sum, it's easy to see that

$$\text{Number squares with side parallel to } n \times n \text{ grid} = \sum_{i=1}^{n-1} i^2 = \frac{n(n-1)(2n-1)}{6}$$

$$\text{Number squares with side parallel to } 19 \times 19 \text{ grid} = \sum_{i=1}^{18} i^2 = \frac{19 \cdot 18 \cdot 37}{6} = \underline{2109}\,.$$

This is the solution to Puzzle #3. Computing all squares of the grid requires recognizing that squares whose sides are parallel to the grid actually induce many (rotated) squares within them. For example:

- There are 18 valid squares (formed using 4 coordinates) that can be inscribed in the square of side-length 18, described above,
- There are 17 valid squares (formed using 4 coordinates) that can be inscribed in each of the 4 squares of side-length 17, described above

- …
- A square of side-length 1 will not contain valid rotated squares inside of it. Therefore, there is only 1 valid square that can be inscribed in a square of side-length 1

Expanding the sum, it's easy to see that

$$\text{Total number squares in an } n \times n \text{ grid } = \sum_{i=1}^{n-1} i^2(n - i) = \frac{n^2(n-1)(n+1)}{12}$$

$$\text{Total number squares in an } 19 \times 19 \text{ grid } = \sum_{i=1}^{18} i^2(n - i) = \frac{361 \cdot 18 \cdot 20}{12} = \underline{10830}\,.$$

This is the solution to Puzzle #1. Computing the number of squares with integer side-lengths is harder, since it requires reasoning about, e.g., pythagorean triangles, and there isn't a clean closed form solution to Puzzle #2 in the general case, as far as I am aware. In the case of a 19×19 grid, one might recognize that it is possible to align four 3-4-5 pythagorean triangles in order to form a square with side-length 5, and this square may be inscribed in a square of side-length 7 whose sides are aligned with the grid's axis. In fact, there are

- 2 different squares with side-length 5 can be inscribed in each square of side-length 7,
- 2 different squares with side-length 10 can be inscribed in each square of side-length 14,
- 2 different squares with side-length 13 can be inscribed in each square of side-length 17,

where the larger squares above have sides aligned with the grid. This allows for computing the total number of squares with integer side-lengths by incorporating an extra term in the sum

$$\text{Number squares with integer side-lengths in a } 19 \times 19 \text{ grid}$$
$$= \sum_{i=1}^{18} i^2 + \sum_{i \in \{7,\, 14,\, 17\}} 2(19 - i)^2 = \underline{2455}\,.$$

This is the solution to Puzzle #2.