Border Line
May 28, 2025 - 2:56pm
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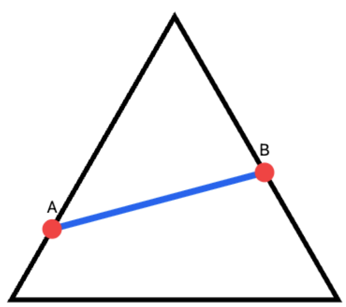
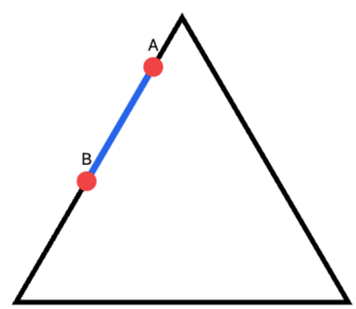
Choose 2 points uniformly on the boundary of a unit equilateral triangle.

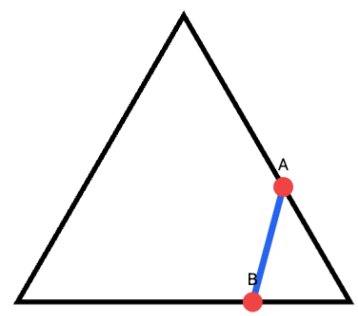
Question: What is the expected value of the distance between the two points?



Length $\overline{AB}$: 0.67 **Side length:** 1.0



Length $\overline{AB}$: 0.40 **Side length:** 1.0



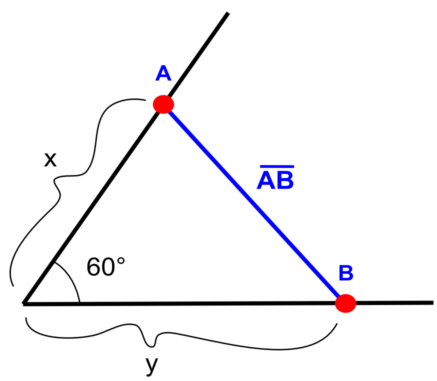
Length $\overline{AB}$: 0.36 **Side length:** 1.0

Expected Length $\overline{AB}$: ______________

Express your answer to **8 decimal places**.

Solution:

First, consider the case where A and B appear on different edges of the triangle. In this case, there is a vertex that connects the faces, and the locations of A and B are effectively determined by two uniformly distributed random variables $0 \le x,\ y \le 1$.



The expected length of $\overline{AB}$ is:

$$E[\ length\ of\ \overline{AB}\ ]\ =\ \int_0^1\int_0^1\ (\ length\ of\ \ \overline{AB}\ |\ x,\ y\ )\cdot pdf(\ x,\ y)\ \ dx\,dy.$$

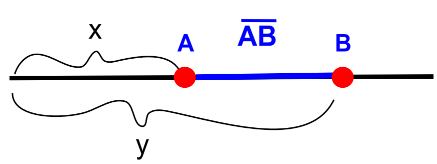
Here, the probability density function $pdf(\ x,\ y)\ =\ 1$ since both x and y are uniform. The length of $\overline{AB}$ can be computed using the Law of Cosines:

$$\begin{aligned}(\ length\ of\ \overline{AB}\ |\ x,\ y\ ) &= \sqrt{x^2 + y^2 - 2xy\cos(60°)} \\ &= \sqrt{x^2 + y^2 - xy}\,.\end{aligned}$$

Now

$$E[\ length\ of\ \overline{AB}\ ]\ =\ \int_0^1\int_0^1\ \sqrt{x^2 + y^2 - xy}\ \ dx\,dy.$$

Next consider the case where A and B appear on the same edge. We can again describe their distance using two uniformly distributed random variables



where again $pdf(\ x,\ y)$ is 1 everywhere and now

$$(\ length\ of\ \overline{AB} \ | \ x, \ y \ ) \ = \ |x \ - \ y|,$$

$$E[\ length\ of\ \overline{AB} \ ] \ = \ \int_0^1\int_0^1 \ |x \ - \ y| \ \ dx\,dy.$$

To compute the true expected length of $\overline{AB}$, sum the expected lengths in the two cases, weighted by the probability of each case. There is a ⅓ chance that A and B appear on the same edges the and a ⅔ chance that A and B appear on different edges[1]. Therefore

$$E[\ length\ of\ \overline{AB} \ ] \ = \ (⅔)\int_0^1\int_0^1 \ \sqrt{x^2 + y^2 - xy} \ \ dx\,dy \ + \ (⅓)\int_0^1\int_0^1 \ |x \ - \ y| \ \ dx\,dy$$

$$\approx \ 0.51643538.$$

This is the solution to the puzzle.

[1] To see this, imagine choosing the point's locations one at a time. The placement of the first point is arbitrary and can be placed anywhere on the triangle's perimeter. When placing the second point, there is a ⅓ chance it will land on the same edge as the first, and a ⅔ chance it will land on a different one.