Line Drive

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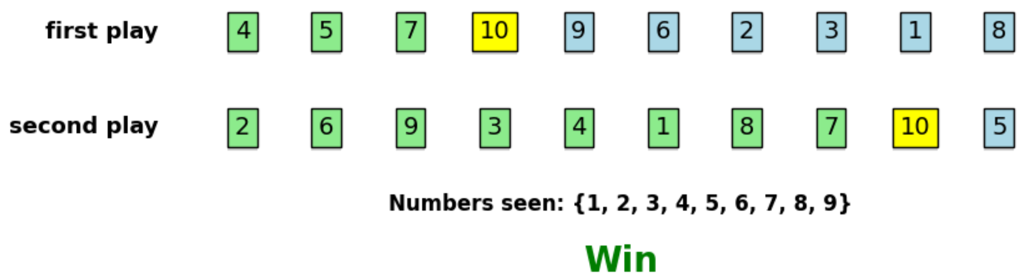
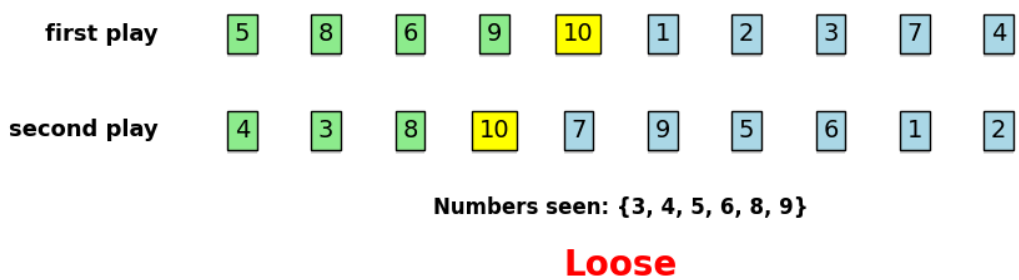
You have 10 cards, each labeled with a number 1-10.Shuffle them, then lay them out in a line. Take a piece of paper and write down all the numbers that appear before the 10.





What is the probability you see all 9 numbers after two plays? ______

What is the probability you see all 9 numbers after four plays? ______

Solution

Part 1. The probability of winning in two plays is $\frac{7,381}{25,200} \approx 0.292897$.

Part 2. The probability of winning in four plays is $\frac{21,945,415,349}{32,006,016,000} \approx 0.685665$.

Explanation

Winning on the first play requires the ten card to appear in the last position (1 / 10 chance).

In the instance where this doesn't happen, and the 10 card appears in a different position, the player will only win on the second play if the ten appears in the last position relative to the remaining cards.

For example, if in the first play the ten appears in the third to last position then there are two cards that need to be found in the second play, and the probability of winning is the probability of the ten appearing in the last position relative to those cards, i.e. $1/3$.

first play:	9	8	3	5	1	4	7	10	6	2
second play:	3	8	2	1	7	6	10	4	9	5

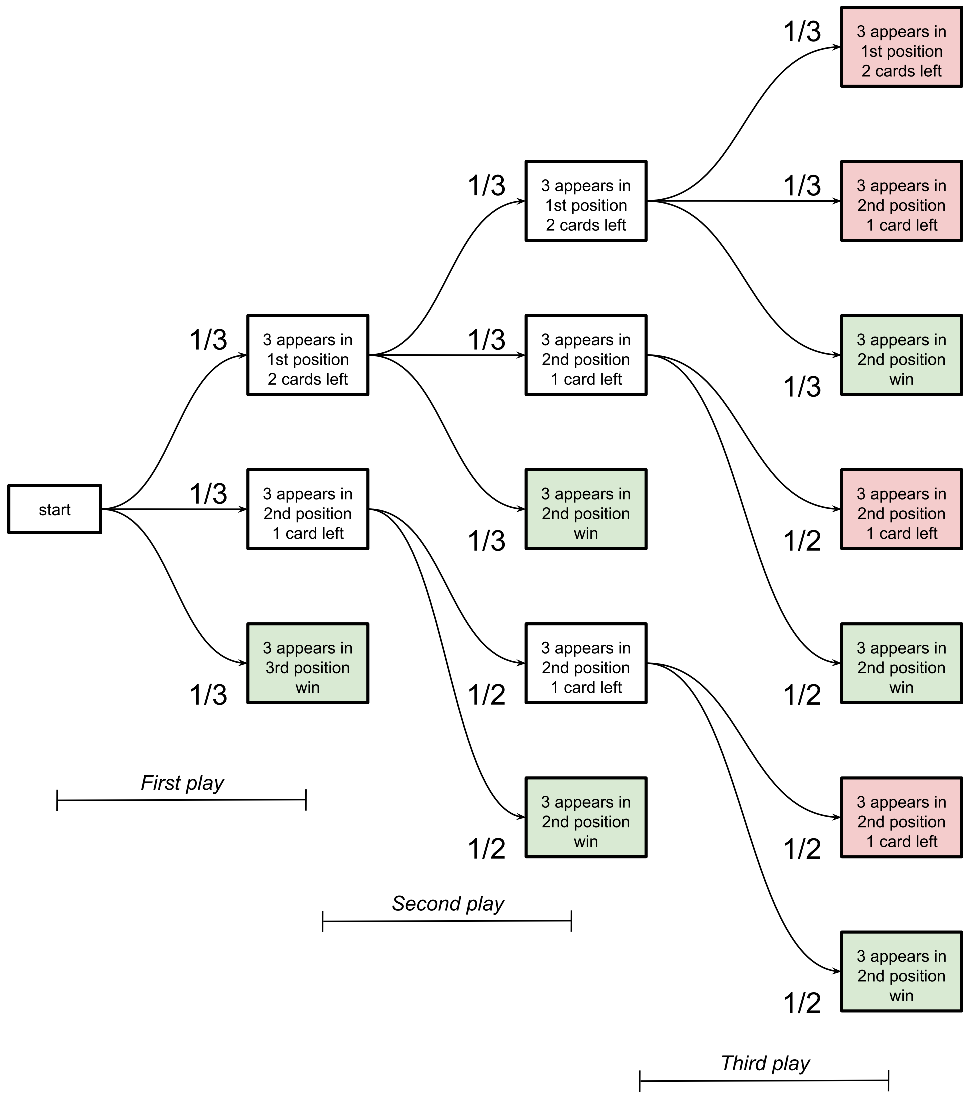
Win if the 10 comes after the 6 and the 2 - ⅓ chance

If instead the ten card appears in the fourth to last position on the first play, then there are three remaining cards that need to be found in the second play, and the probability of winning is $1/4$. Each of the ten possible places where the ten card could appear are equally likely, and aggregating we find that the probability of winning in two plays is $\frac{1}{10}\left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{10}\right)$ or

$$Prob[\, win\ in\ 2\ plays\,] = \frac{1}{10}\sum_{n=1}^{10}\frac{1}{n} = \frac{7,381}{25,200} \approx 0.292897$$

This is the solution to the first part.

If instead, on the second play, the ten card does not come in the last position relative to the remaining cards from the first play, then the cards which came after the 10 card in the second play must come before the 10 on the third play. See the following figure, which demonstrates how the pattern continues, in the reduced case of 3 cards



Example with three cards and three plays. The probability of winning is $\frac{1}{3}\left(\frac{1}{1}+\frac{1}{2}\left(\frac{1}{1}+\frac{1}{2}\right)+\frac{1}{3}\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}\right)\right)$.

Continuing this pattern, the probability of winning in three plays with 10 cards is

$$Prob[\,win\ in\ 3\ plays\,] = \frac{1}{10}\sum_{n=1}^{10}\frac{1}{n}\sum_{m=1}^{n}\frac{1}{m} = \frac{32{,}160{,}403}{63{,}504{,}000} \approx 0.506431$$

and the probability of winning in of four plays with 10 cards is

$$Prob[\,win\ in\ 4\ plays\,] = \frac{1}{10}\sum_{n=1}^{10}\frac{1}{n}\sum_{m=1}^{n}\frac{1}{m}\sum_{l=1}^{m}\frac{1}{l} = \frac{21{,}945{,}415{,}349}{32{,}006{,}016{,}000} \approx 0.685665.$$

This is the solution to the second part.